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Analysis of Whisker-Toughened Ceramic Components Using an Interactive Reliability Model

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Realizing wider use of ceramic matrix composites requires the development of advanced structural analysis technologies. This article focuses on the use of interactive reliability models to predict component probability of failure. The deterministic Willam-Warnke failure criterion is the theoretical basis for the reliability model presented herein. The model has been incorporated into a test-bed software program. This computer program is coupled to a general-purpose finite element program. A simple structural problem is presented to illustrate the reliability model and the computer algorithm.

Introduction

CERAMIC matrix composites (CMC) encompass a wide range of material systems. Of interest here are the particulate reinforced ceramics, specifically where the reinforcing particles are short fibers or whiskers. This class of ceramic composites has been used successfully as wear resistant parts, cutting tools, and seals. They have the potential for competing with metals in many demanding applications. The use of ceramic-based material systems in structural components offers highly advantageous improvements in certain mechanical properties. These improvements focus on increases in strength and stiffness, in addition to the increased creep and corrosion resistance commonly ascribed to ceramic materials. All of these traits are maintained even at very high service temperatures. Considering that ceramics are produced from abundant nonstrategic materials, it is not surprising that research has focused on improving ceramic material properties through processing, as well as on establishing protocols for sound design methods.

Unfortunately, CMC material systems have several mechanical characteristics that must be carefully considered when designing structural components. The most deleterious of these characteristics are low strain tolerance, low fracture toughness, and a large variation in failure strength. Thus, analyses of components fabricated from ceramic materials require a departure from the usual deterministic design philosophy (i.e., the factor of safety approach) prevalent in the analysis of metallic structural components. Even though improved processing techniques have resulted in fewer inhomogeneities, more uniform whisker distributions, and denser matrices, failure for these discrete-particle-toughened ceramics remains a stochastic process. Because failure is dominated by the scatter in strength, statistical design approaches must be used.

In predicting the failure behavior of composite materials, a general philosophical division separates analytical schools of thought into microstructural and phenomenological methods. Blass and Ruggles[1] point out that analysts who use microstructural methods would design the material, and examples of this type of modeling include the works of Lange,[2] Faber and Evans,[3] and Wetherhold.[4] Each work focuses on different crack-mitigation processes (e.g., crack bowing, crack tilting and twisting, bridging, etc.). The primary intent of models that evolve from this method is to develop predictive tools for optimizing material properties by engineering the microstructure. Supposedly, these models could also be used to design structural components. However, these crack-mitigation processes strongly interact, and it is difficult to detect experimentally or to predict analytically which sequence of mechanisms leads to failure. This precludes adopting a microstructural approach in analyzing components fabricated from whisker-toughened CMC materials.

In the alternative approach, failure behavior is analyzed in terms of macrovariables through the use of phenomenological methods. The material element under consideration is small enough to be homogeneous in stress and temperature yet large enough to contain enough whiskers to provide an element that is a statistically homogeneous continuum. The analyst then designs the material by taking full advantage of the beneficial properties and compensating in some rational manner for the deleterious ones. Duffy et al.[5] presented an overview of phenomenological methods that incorporate noninteractive reliability models. In addition to capturing any inherent scatter in strength, these noninteractive models account for material symmetry imposed by the reinforcement. Depending on fabrication, a whisker-toughened composite may have local material symmetries that are isotropic, transversely isotropic, or orthotropic. In this paper we focus on whisker-toughened CMC materials that are isotropic and examine the viability of using an interactive reliability model for the structural analysis of components fabricated from this material. In the final section the authors indicate how to extend this analytical approach to account for anisotropic material symmetries.

Failure Criterion

An essential step in developing an interactive reliability model requires formulating a deterministic failure criterion that reflects the limit state behavior of the material. Miki et al.[6] and de Roo and Paluch[7] have adopted this approach in computing the reliability of unidirectional composites. In both articles the Tsai-Wu failure criterion is adopted. Different failure behavior is allowed in tension and compression, both in the fiber direction and transverse to the fiber direction. Here, it is assumed that, for isotropic whisker-toughened ceramics, failure behavior exhibits both a lower tensile strength than the material's compressive strength and a dependence on the hydrostatic component of the stress state. This behavior has not been documented experimentally in the open literature for whisker-toughened ceramics, but this type of response

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has been exhibited by monolithic ceramics (see Adams,[8] Ikeda and Igaki,[9] and Ikeda et al.[10]). The authors assume that, qualitatively, the failure behavior of isotropic whisker-toughened ceramics mimics the failure behavior of monolithic ceramics.

In general, a failure criterion (or limit state) defines the conditions under which a structural component can no longer fulfill its design function. Let

$$\tilde{Y} = (Y_1, Y_2, \ldots, Y_n) \quad (1)$$

denote a vector of design variables (e.g., strength parameters, cyclic load limits, allowable deformation at critical locations of a component, etc.). A limit state function $f(\tilde{Y})$, which stipulates how the design variables interact when failure occurs, defines a surface in the n-dimensional design variable space. Typically, the following simple expression

$$f(\tilde{Y}) = 0 \quad (2)$$

is used to mathematically define the failure surface. A design point for a structural component that falls within the surface defines a successful operational state. This region is denoted ω_s. If the design point lies on the surface (denoted ω_f), the component fails. Note that points outside a deterministic failure surface represent inaccessible limit states.

Here, the design variable space is restricted to material strength parameters. These material strength parameters are closely associated with components of the stress tensor; hence, in general,

$$f = f(\tilde{Y}, \sigma_{ij}) \quad (3)$$

where σ_{ij} represents the Cauchy stress tensor. Unless noted otherwise, the implied range of the tensor indices is from 1 to 3. Because of the dependence on the stress tensor, phenomenological failure criteria must be formulated as invariant functions of σ_{ij}. Using this approach makes a criterion independent of the coordinate system used to define the stress tensor (i.e., the criterion exhibits frame indifference). For isotropic materials the failure function can be expressed as

$$f(\tilde{Y}, I_1, J_2, J_3) = 0 \quad (4)$$

which guarantees that the function is form invariant under all proper orthogonal transformations. Here, I_1 is the first invariant of the Cauchy stress σ_{ij}, J_2 is the second invariant of the deviatoric stress S_{ij}, and J_3 is the third invariant of the deviatoric stress. These quantities are defined in the following manner:

$$S_{ij} = \sigma_{ij} - (\tfrac{1}{3})\delta_{ij}\sigma_{kk} \quad (5)$$

$$I_1 = \sigma_{kk} \quad (6)$$

$$J_2 = (\tfrac{1}{2})S_{ij}S_{ji} \quad (7)$$

$$J_3 = (\tfrac{1}{3})S_{ij}S_{jk}S_{ki} \quad (8)$$

where δ_{ij} is the identity tensor. Admitting I_1 to the functional dependence allows for a dependence on hydrostatic stress. Admitting J_3 (which changes sign when the direction of a stress component is reversed) allows different behavior in tension and compression to be modeled with a single unified failure function.

Two representative failure criteria formulated by Willam and Warnke[11] and Ottosen[12] satisfy the requisite failure behaviors of reduced tensile strength and sensitivity to hydrostatic stress. The Willam-Warnke criterion is expressed as

$$f = \lambda\left(\frac{\sqrt{J_2}}{Y_C}\right) + B\left(\frac{I_1}{Y_C}\right) - 1 \quad (9)$$

where

$$B = B(Y_T, Y_C, Y_{BC}) \quad (10)$$

and

$$\lambda = \lambda(Y_T, Y_C, Y_{BC}, J_3) \quad (11)$$

where Y_T denotes the tensile strength of the material, Y_C is the compressive strength, and Y_{BC} represents the equal biaxial compressive strength of the material. This model is termed a three-parameter model, referring to the three material strength parameters (Y_T, Y_C, and Y_{BC}) used to characterize the model. Failure is defined when $f = 0$ and the multiaxial criterion is completely defined in all regions of the stress space. In comparison, the Ottosen criterion introduces more flexibility by including an additional parameter. The four-parameter Ottosen criterion retains all of the characteristics of the Willam-Warnke criterion; however, the limit state function is quadratic in $\sqrt{J_2}$. Since the Ottosen criterion requires characterizing an additional model parameter, the interactive reliability model will be studied in terms of the simpler Willam-Warnke criterion.

Since the failure function is dependent on the six components of the Cauchy stress tensor, a stress state and its relative position to the failure surface can be depicted graphically in various ways. Graphical representation can take place in a two- or three-dimensional stress space using components of the Cauchy stress tensor as coordinate axes. However, the function and the physical implications associated with the function can be conveniently viewed in the three-dimensional stress space where the principal stresses serve as orthogonal coordinate axes. This is known as the Haigh-Westergard stress space. In Fig. 1 the function is projected on a two-dimensional plane that is defined by the σ_1-σ_2 axes. This plane is contained in the Haigh-Westergard space. The model parameters have been arbitrarily chosen such that $Y_T = 0.2$ GPa, $Y_C = 2$ GPa, and $Y_{BC} = 2.32$ GPa. Note the tensile strength is an order of magnitude lower than the compression strength. The ratio of intercepts along the tensile and compressive axes (i.e., the ratio of Y_T to Y_C) reflects this. Alternatively, the function can be projected on the Π plane. In Fig. 2 several failure surfaces are presented that coincide with negatively increasing values of the hydrostatic component of the stress state (I_1). Finally, Fig. 3 depicts the failure function projected onto the I_1-$\sqrt{J_2}$ coordinate space. Again, this is a two-dimensional plane contained in the Haigh-Westergard stress space. The key feature is that the Willam-Warnke failure function is a linear function

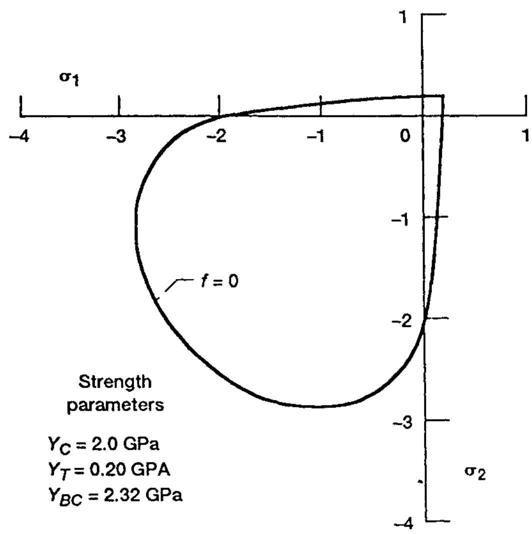

Fig. 1 Willam-Warnke failure function projected onto the σ_1-σ_2 stress space.

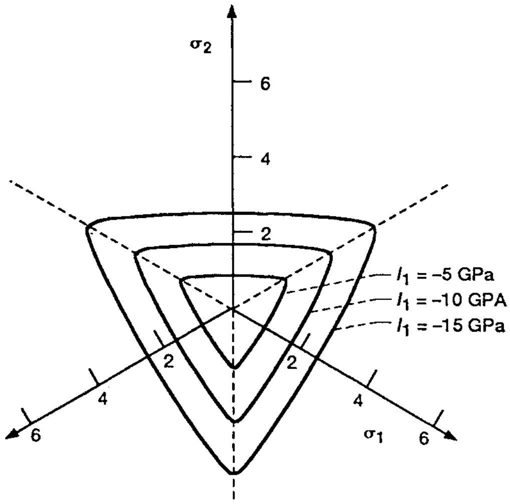

Fig. 2 Willam-Warnke failure function depicted in the deviatoric stress space for values of $I_1 = -5, -10$, and -15.

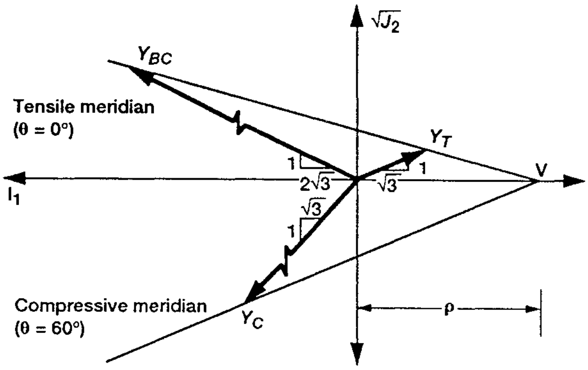

Fig. 3 Willam-Warnke failure function depicted in the I_1-$\sqrt{J_2}$ coordinate space.

in $\sqrt{J_2}$. Also note that the load paths used to determine the three strength parameters (Y_C, Y_T, and Y_{BC}) are depicted in Fig. 3. As a matter of convenience and for illustration, the figure is not drawn to scale.

Interactive Reliability Model

Adopting the Willam-Warnke phenomenological failure criterion described in the previous section, the general functional dependence of the limit state function is

$$f(\tilde{Y}, \sigma_{ij}) = f(Y_T, Y_C, Y_{BC}, I_1, J_2, J_3) \tag{12}$$

where the specific form of $f(\tilde{Y}, \sigma_{ij})$ is given by Eq. (9). It is assumed that the material strength parameters are independent random variables. The objective is to compute the reliability (denoted by $\mathbb{R}$) of a material element of unit volume, given a stress state, where the components of the stress tensor are considered to be deterministic parameters. This philosophy, i.e., that material strength parameters are independent random variables and the load parameters are deterministic, is not without precedent. The reader is directed to the overview by Duffy et al.[13] for references. Conceptually, both strength and load can be treated as random variables in the analysis that follows; however, the level of variation in material strength is often on the order of 100% (characterized as a percentage increase in the smallest strength value in a data set) for ceramic materials. Thus, the authors focus on material strength as the random variable of primary interest.

The reliability of an element of unit volume is given by the following expression:

$$\mathbb{R} = \text{Probability}[f(\tilde{Y}) < 0] \tag{13}$$

It is assumed that the element is homogeneous in stress, such that stress gradients are not present in the element. Under the assumption that the random variables are independent, this probability is given by the product of the marginal probability density functions integrated over the region ω_s, which is defined by Eq. (9), i.e.,

$$\mathbb{R} = \iint_{\omega_s}\int p_1(y_C)p_2(y_T)p_3(y_{BC})\, dy_C\, dy_T\, dy_{BC} \tag{14}$$

where $p_1(y_C)$, $p_2(y_T)$, and $p_3(y_{BC})$ are the marginal probability density functions of the random variables representing the material strength parameters. An attractive feature of the analytical approach adopted here is the flexibility of using different formulations of the marginal probability density function, such as a three-parameter Weibull distribution, a log-normal distribution, or some other type of distribution. The appropriate distribution function would be dictated by the experimental failure data for a particular strength parameter. Here, the Weibull formulation is considered where the three-parameter probability density function for a continuous random variable x is given by the expression

$$p(x) = \left(\frac{\alpha}{\beta}\right)\left(\frac{x-\gamma}{\beta}\right)^{(\alpha-1)} \exp\left[-\left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right] \tag{15}$$

for $x > \gamma$, and

$$p(x) = 0 \tag{16}$$

for $x \le \gamma$. The cumulative distribution function is given by

$$P(x) = 1 - \exp\left[-\left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right] \tag{17}$$

for $x > \gamma$, and

$$P(x) = 0 \tag{18}$$

for $x \le \gamma$. In Eqs. (15) and (17) α (restricted to positive values) is the Weibull modulus (or the shape parameter), β (similarly restricted to positive values) is the Weibull scale parameter, and γ is a threshold parameter. If the applied stress is lower than this threshold parameter, the probability of component failure is defined as zero. Here, a conservative approach is adopted such that the value of the threshold parameter is taken to be zero. This yields the more widely used two-parameter Weibull formulation for the random strength parameters. Each random variable is associated with its own set of Weibull parameters, i.e., Y_C and (α_C, β_C), Y_T and (α_T, β_T), as well as Y_{BC} and $(\alpha_{BC}, \beta_{BC})$.

Explicit integration of Eq. (14) is intractable because of the form of the limit state function, which defines the integration domain ω_s. (See Sun and Yamada[14] and Wetherhold[15] for the details of this type of integration for simpler interactive failure criteria.) Herein, Monte Carlo simulation is used to numerically evaluate the triple integral. This technique involves generating a uniform random sample of size k for each of the random variables. For a given stress state, the failure function is computed for each sample of random variables. If $f(\tilde{Y}) < 0$ for a given trial, then that trial is recorded as a success. By repeating this process a suitable number of times for a given state of stress, a cumulative distribution for the element is generated. For a sufficiently large sample size, reliability can be computed as

$$\mathbb{R} = n/k \tag{19}$$

where n is the number of successful trials [i.e., $f(\tilde{Y}) < 0$].

This technique was used in calculating the reliability contours depicted in Fig. 4. The reliability contours represent a

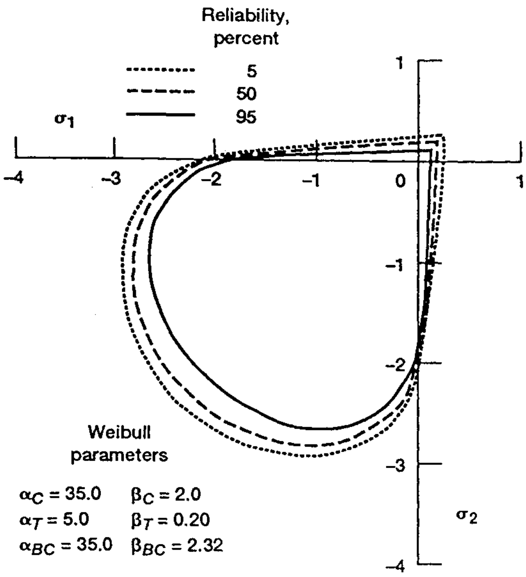

Fig. 4 Reliability contours using the reliability model projected onto the σ_1-σ_2 stress space.

homogeneously stressed material element of unit volume, and for dimensionless $\mathbb{R}$ the Weibull parameter β has units of stress times (volume)$^{1/\alpha}$. The Weibull parameters associated with the tensile strength random variable are arbitrarily chosen, specifically $\alpha_T = 5$ and $\beta_T = 0.2$. Similarly, the Weibull parameters associated with the compressive strength random variable are arbitrarily specified, with $\alpha_C = 35$ and $\beta_C = 2$. Finally, the Weibull parameters associated with the equal biaxial compressive strength random variable are $\alpha_{BC} = 35$ and $\beta_{BC} = 2.32$. The three surfaces depicted in Fig. 4 correspond to $\mathbb{R} = 0.05$, 0.5, and 0.95. Note that the reliability contours retain the general behavior of the deterministic failure surface from which they were generated. In general, as the various α increase, the spacings between contours diminish. Eventually the contours would not be distinct, and they would effectively map out a deterministic failure surface. Finally, note that an increase in the various β would shift the relative position of the contours outward.

Component Reliability Analysis

In the previous section the details for computing the reliability of a single element were outlined, assuming a homogeneous state of stress and a unit volume. Typically, for a structural component with a varying stress field, the component is discretized, and the stress field is characterized using finite element methods. Since component failure may initiate in any of the discrete elements, it is useful to consider a component from a systems viewpoint. A discretized component is a series system if it fails when one of the discrete elements fails. This approach gives rise to weakest-link reliability theories. A discretized component is a parallel system when the failure of a single element does not necessarily cause the component to fail, since the remaining elements may sustain the load through redistribution. Parallel systems lead to what has been referred to in the literature as "bundle theories." Since it is assumed that, qualitatively, the failure behavior of whisker-toughened ceramics mimics monolithic ceramics, we adopt a weakest-link reliability theory for designing structural components.

If the failure of an individual element is considered a statistical event and if these events are assumed to be independent, then the probability of failure of a discretized component is given as

$$P_f = 1 - \prod_{\nu=1}^{N} R_\nu \tag{20}$$

where N is the number of discrete finite elements for a given component. For this computational procedure it is necessary to compute the reliability of the νth discrete element (R_ν) in the following manner. Recall that $\mathbb{R}$ (the reliability based on a unit volume) is defined by Eqs. (13) and (14) but calculated using the Monte Carlo techniques. These same techniques can be used to compute R_ν if the Weibull scale parameters are adjusted to reflect the size of the element. In general, each scale parameter (β_T, β_C, β_{BC}) is adjusted using the following transformation:

$$\beta^* = \beta \left(\frac{1}{V_\nu}\right)^{1/\alpha} \tag{21}$$

where V_ν is the volume of the νth element, and β^* is the adjusted scale parameter. It is assumed that the threshold parameter for each random strength variable is zero. No adjustment is necessary for the Weibull moduli. In the preceding discussion on reliability models it is implied that the failure of whisker-toughened CMC originates from volume flaws. However, it is quite possible that component failure may be caused by surface and volume flaws, i.e., that competing failure modes may exist. These competing failure modes are independent and would have distinctly different Weibull parameters that characterize the marginal probability density functions. Accordingly, the transformation defined by Eq. (21) can be used for surface flaw analyses if V_ν is replaced by the area of the νth element, A_ν. However, for brevity the authors have only focused on volume flaw analyses in this article.

The numerical procedure just discussed has been incorporated in the public domain computer algorithm called toughened ceramics analysis and reliability evaluation of structures (TCARES). Currently, this algorithm is coupled to the MSC/NASTRAN finite element code (see Duffy et al.[5] for a complete description of the algorithm). Before one can use TCARES, the interactive reliability model presented here must be characterized using an extensive database that includes multiaxial experiments. We feel strongly that it is not sufficient to simply characterize the Weibull parameters for each random strength variable but that multiaxial experiments should be conducted to assess the accuracy of the interactive modeling approach. However, once the Weibull parameters have been characterized for each random strength variable, the algorithm allows the design engineer to predict the reliability of a structural component subject to quasistatic multiaxial load conditions. For the application that follows, isothermal conditions are considered. However, the algorithm is capable of nonisothermal analyses if the Weibull parameters are specified at a sufficient number of temperature values. To illustrate certain aspects of the interactive model and the TCARES algorithm, we perform a reliability analysis on a Brazilian disk test specimen.

The Brazilian disk is a multiaxial test specimen and is used to circumvent the alignment difficulties encountered in tensile testing brittle materials. In addition, the Brazilian disk has been used to determine tensile strengths of brittle materials that exhibit lower tensile strengths than their compressive strength, e.g., concrete and rock. The analytical details concerning the stress field of the Brazilian disk have been discussed by a number of authors including Hondros,[16] Vardar and Finnie,[17] Chong et al.,[18] and Fessler and Fricker.[19] Tensile failure usually occurs along the diameter directly beneath the applied load (see Fig. 5a), splitting the disk in two. However, the region of the disk directly beneath the load experiences very large compressive stress states that dissipate slowly. The interactive model presented here allows for a reduction in reliability when compressive stress states are present. Thus, the Brazilian disk is used to compare the interactive model with other widely used reliability models that do not account for compressive stress states. For simplicity we compare the

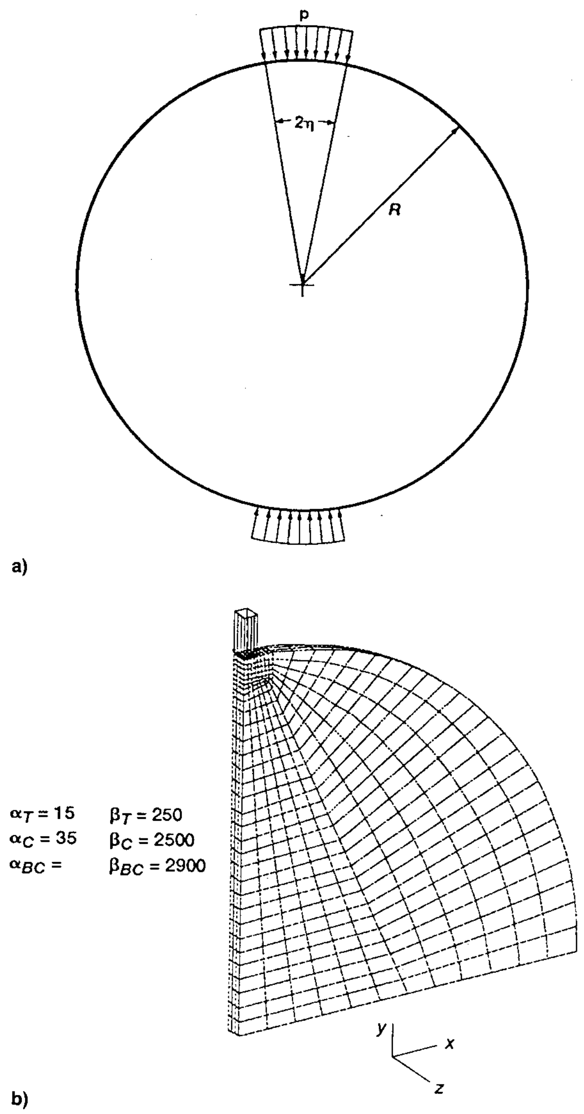

Fig. 5 Brazilian disk test specimen: a) geometry and pressure load, b) finite element model.

interactive model with the principle of independent action (PIA) reliability model.

It is assumed that the disk is fabricated from an isotropic whisker-toughened CMC material with a Young's modulus of 300 GPa and a Poisson ratio of 0.2. A compressive pressure loading of 1000 MPa was applied to the disk such that the subtended angle η is 0.039 rad. The Weibull parameters associated with each random strength variable were arbitrarily chosen. Specifically, the Weibull parameters associated with the tensile strength random variable are $\alpha_T = 15$ and $\beta_T = 250$. The Weibull parameters associated with the compressive strength random variable are $\alpha_C = 35$ and $\beta_C = 2500$. Similarly, the Weibull parameters associated with the equal biaxial strength random variable are $\alpha_{BC} = 35$ and $\beta_{BC} = 2900$. Note that the β parameters have units of megapascals per millimeter$^{3/\alpha}$. The disk, which has a radius of 50 cm and a thickness of 5.0 cm, was modeled using $\frac{1}{8}$ symmetry with 1044 finite elements (Fig. 5b). The elements were eight-node bricks elements (MSC/NASTRAN HEX/8). The tensile stress in the x direction near the center of the disk was 24.8 MPa (Fig. 6). This stress remains fairly constant along the vertical diameter, except in the near vicinity of the load, where this stress component changes sign and becomes compressive. The elements near the load experience large compressive stresses (~997 MPa) in the y direction that similarly dissipate slowly down the diameter (see Fig. 7).

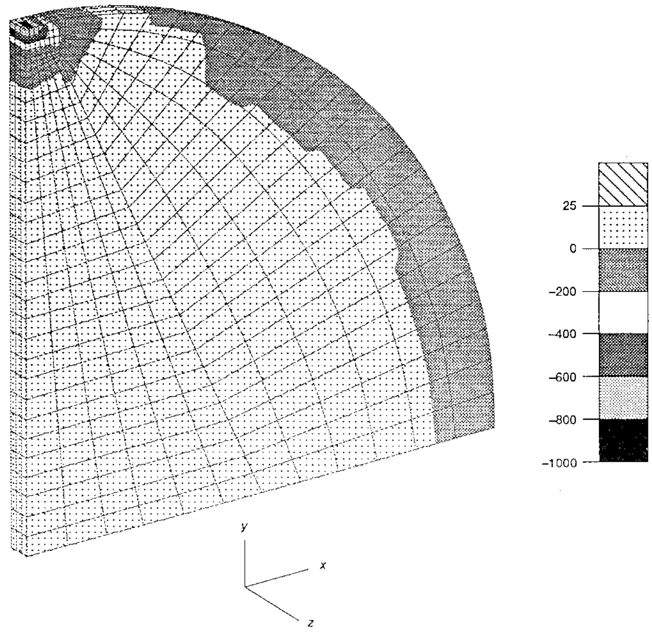

Fig. 6 Distribution of normal stresses in the x direction.

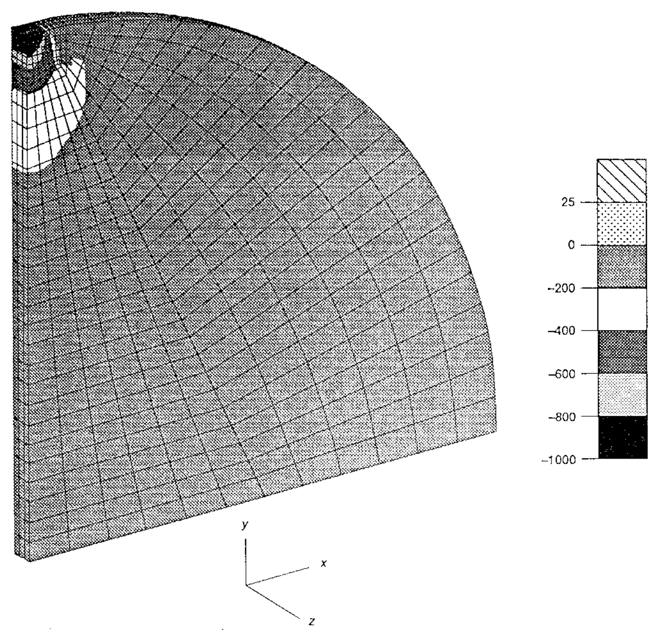

Fig. 7 Distribution of normal stresses in the y direction.

When this particular discretized component was analyzed using the PIA model (with $\alpha = 15$ and $\beta = 250$), the component reliability was 99.9%. Note that compressive stress states (specifically compressive principal stresses) do not affect component reliability when using the PIA model. This assumption is similarly adopted for other more popular reliability theories such as the model proposed by Batdorf and Crose.[20] This lack of accounting for compressive stress states may be a nonconservative assumption depending on the values of the Weibull parameter that characterize the compressive strength random variables. Analyzing the disk using the interactive reliability model (and the Weibull parameters stipulated earlier) resulted in a component reliability of 77.7%. Again, note that the Weibull parameters for the compressive strength random variables were arbitrarily specified. However, α values of 35 approach values for metals that have deterministic strength parameters, and an increase in the β values of over an order of magnitude relative to β_T represents conservative estimates of these Weibull parameters. Thus, accounting for compressive stress states may play an important role in the analysis of certain structural components.

Discussion

The basic features of conducting a reliability analysis using an interactive failure criterion have been illustrated. The deterministic Willam-Warnke failure criterion serves as the theoretical basis for the reliability model presented here. The model has been implemented into TCARES, a test-bed software program. Since this algorithm has been coupled with a general-purpose finite element program, design engineers may use the code to predict the time-independent reliability of a structural component subject to quasistatic multiaxial load conditions. A simple structural problem was presented to illustrate the reliability model and the computer algorithm.

In addition the authors wish to briefly summarize how to extend this type of interactive reliability model to account for material anisotropy. Using orthotropic materials as an example, the failure function must reflect the stress state (as was done in this article) and the appropriate material symmetry. For orthotropy this requires that

$$f = f(\tilde{Y}, \sigma_{ij}, a_i, b_i) \tag{22}$$

where a_i and b_i are orthogonal unit vectors that represent the local orthotropic material directions. Because f is a scalar function, it must remain form invariant under arbitrary proper orthogonal transformations. Work by Reiner,[21] Rivlin and Smith,[22] Spencer,[23] and others demonstrated that, by applying the Cayley-Hamilton theorem and the elementary properties of tensors, a finite set of invariants (known as an integrity basis) can be derived for any scalar function that is dependent on first- and second-order tensor quantities. Form invariance of the scalar function is insured if the function is constructed using invariants that constitute the integrity basis. A number of authors have used this methodology to develop scalar valued functions that are dependent on stress (a second-order tensor) and material directions (usually characterized by first-order tensors). Lance and Robinson[24] proposed a maximum shear stress plasticity theory for composites. Boehler and Sawczuk[25] proposed a plasticity theory for anisotropic cohesive materials. Arnold[26] developed a thermoelastic constitutive theory for transversely isotropic materials. Finally, Robinson and Duffy[27] developed a viscoplastic constitutive theory for transversely isotropic materials. Clearly, the future direction alluded to here is not without precedent. However, for anisotropic whisker-toughened ceramic composites, the failure function must not only reflect the material anisotropy but also account for reduced tensile strength and a dependence on the hydrostatic component of stress, if this behavior is exhibited experimentally.

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